ON ONE PROBLEM OF STABILITY OF A PIPE WITH A FLUID FLOWING THROUGH IT

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PMM Vol.29, № 4, 1965, pp. 760-762

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(Received May 6, 1964)

The problem of stability of a pipe with a f uid flowing through it, which is examined by Feodos'ev in [1] by the Galerkin method, is solved by a direct Liapunov's method without assuming that the deflection is expressed as a product of a function of coordinates and a function of time. It is shown, that the value of the critical velocity obtained in [1] is exact. It is found, that for nonzero subcritical velocities of the fluid flow the free vibrations of the pipe are running waves along the pipe.

1. After introducing nondimensional variables, equations in [1] for the deflection $w(x,t)$ of the pipe, can be expressed in the form

$$\frac{\partial^4 w}{\partial x^4} + \pi^2 v^2 \frac{\partial^2 w}{\partial x^2} + 2\pi\rho v \frac{\partial^2 w}{\partial x \partial t} + \frac{\partial^2 w}{\partial t^2} = 0 \tag{1.1}$$

$$w(x,t)\Big|_{x=0} = \frac{\partial^2 w(x,t)}{\partial x^2}\Big|_{x=0} = w(x,t)\Big|_{x=1} = \frac{\partial^2 w(x,t)}{\partial x^2}\Big|_{x=1} = 0$$

The nondimensional velocity v of the fluid and the nondimensional mass parameter ρ are determined by the following equations:

$$v = V\frac{a}{\pi}\left(\frac{\rho_2 F_2}{EI}\right)^{1/2}, \qquad \rho = \left(\frac{\rho_2 F_2}{\rho_1 F_1 + \rho_2 F_2}\right)^{1/2}$$

where a is the length of the pipe, EI is the stiffness of the pipe, $\rho_1 F_1$ and $\rho_2 F_2$ are the masses of the pipe and the fluid, correspondingly, per unit length of the pipe, V is the velocity of the fluid in the direction of the x-axis.

It is easy to verify, that the solutions $w(x,t)$ of the problem (1.1) satisfy the relations

$$\frac{dH}{dt} = 0, \qquad H = \int_0^1 dx \left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \overline{w}}{\partial x^2} - \pi^2 v^2 \frac{\partial w}{\partial x}\frac{\partial \overline{w}}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial \overline{w}}{\partial t}\right) \tag{1.2}$$

Here and further on, the functions $w(x,t)$ are assumed to be continuous on the basis of x,t together with the derivatives, which appear when we obtain Expressions (1.2) from Equations (1.1). The third term of the first equation (1.1) did not appear in the energy integral (1.2), which proves the gyroscopic character of this term.

Equations (1.1) allow a solution $w(x,t) = 0$, which corresponds to the equilibrium of the pipe. During the investigation of the stability of this equilibrium let us take as a measure of perturbations the following functional:

$$\rho(z) = \sup_x w\bar{w} + \sup_x \frac{\partial w}{\partial x}\frac{\partial \bar{w}}{\partial x} + \int_0^1 dx \left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \bar{w}}{\partial x^2} + \frac{\partial w}{\partial t}\frac{\partial \bar{w}}{\partial t}\right) \tag{1.3}$$

which is defined at the points

$$z = \left[w(x,t),\ \frac{\partial w(x,t)}{\partial t}\right]_{t=\mathrm{const}}$$

The equilibrium $w(x,t) \equiv 0$ will be considered stable, if for any $\varepsilon > 0$ we can find such $\delta(\varepsilon) > 0$, that any solution of the problem (1.1) which at the initial instant t_0 satisfies the condition

$$\rho(z(t_0)) < \delta \tag{1.4}$$

for all $t \geqslant t_0$ of the region of definition of the solution $w(x,t)$, satisfies the condition

$$\rho(z(t)) < \varepsilon \tag{1.5}$$

Here, we have assumed the definition of stability 3.2. in [2], which is uniform for a set of initial instants $T_0 = T\ (-\infty < t < \infty)$.

2. Let us make use of the functional $H(z)$, defined by the second equation (1.2), for the proof of the following assertion: when the condition

$$v^2 < 1 \tag{2.1}$$

is satisfied, the equilibrium $w(x,t) \equiv 0$ is stable.

The first equation (1.2) shows that for the examined solutions $w(x,t)$ of the problem (1.1) the functional $H(z)$ does not increase with time. According to the theorem 5.2 of [2], it remains to prove, that when condition (2.1) is satisfied, the functional $H(z)$ will be positive-definite and continuous with respect to the degree of perturbations (1.3). These properties follow immediately from the relations

$$H(z) \geqslant {}^1/_8 (1 - v^2)\,\rho(z), \qquad H(z) \leqslant \rho(z)$$

in the proof of which it was taken into account that the following inequalities are satisfied for the functions $w(x,t)$, which describe the motions of the pipe:

$$\int_0^1 dx \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \bar{w}}{\partial x^2} \geqslant \pi^2 \int_0^1 dx \frac{\partial w}{\partial x}\frac{\partial \bar{w}}{\partial x} \geqslant \pi^4 \int_0^1 dx\, w\bar{w}$$

$$\int_0^1 dx \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \bar{w}}{\partial x^2} \geqslant \sup_x \frac{\partial w}{\partial x}\frac{\partial \bar{w}}{\partial x}, \qquad \int_0^1 dx \frac{\partial w}{\partial x}\frac{\partial \bar{w}}{\partial x} \geqslant \sup_x w\bar{w} \tag{2.2}$$

The first two inequalities (2.2) can be obtained by the methods, given in papers [3 and 4], the remaining two inequalities can be obtained easily with the aid of the Cauchy-Buniakovskii inequality.

The assertion which was proved, leaves an open question about the stability of equilibrium $w(x,t) \equiv 0$ when the condition (2.1) is violated.

It is easy to verify, that when any of the following conditions are satisfied

$$v = \pm m \qquad (m = 1, 2, \ldots) \tag{2.3}$$

the equilibrium $w(x,t) \equiv 0$ is unstable, because in this case Equations (1.1) admit the solution

$$w(x,t) = c\,[(t - t_0)\sin m\pi x + \varphi(x)] \tag{2.4}$$

$$\varphi(x) = \pm \frac{2\rho}{m^2\pi^2}\left\{x\,[1 - (-1)^m] - 1 + \cos m\pi x + \frac{m\pi x}{\pi}\sin m\pi x\right\}$$

The choice of an arbitray constant c in the solution (2.4) allows to satisfy the condition (1.4) at the initial instant t_0 for any given number $\delta > 0$; nevertheless, the condition (1.5) will be violated for sufficiently large $t > t_0$. When anyone of the conditions (2.3) is satisfied, the instability of equilibrium $w(x,t) \equiv 0$ is stipulated by the growth of the solution (2.4) in time.

So the exact value of the first nondimensional critical velocity, suitable for Equations (1.1), is given by the equality $v = \pm 1$. Hence, by changing to the dimensional quantities we obtain the critical velocity [1]

$$V = \pm \frac{\pi}{a}\left(\frac{EI}{\rho_2 F_2}\right)^{1/2}$$

3. The condition of stability (2.1) is obtained without any limitation of the form of the solutions of the problem (1.1). Introducing a complementary assumption on the representation of the solution in the form

$$w(x,t) = X(x)\, e^{\omega t} \tag{3.1}$$

from (1.1) we obtain for the function $X(x)$ and for the frequency $\omega = p + iq$ the generalized boundary value problem

$$X^{IV}(x) + \pi^2 v^2 X''(x) + 2\pi\rho v\omega X'(x) + \omega^2 X(x) = 0 \tag{3.2}$$

$$X(0) = X''(0) = X(1) = X''(1) = 0$$

The zeros of the eigenfunction $X(x)$, which correspond to some eigenvalue ω, divide the interval $0 \leqslant x \leqslant 1$ in a finite number of intervals $\xi_1, \xi_2, \ldots, \xi_n$, in each of these intervals the function can be expressed in the form

$$X(x) = |X(x)|\, e^{i\psi(x)}$$

where $\psi(x) = \operatorname{Arg} X(x)$ is continuous in the interval ξ_k. From here we can see that the real and imaginary parts of the solution (3.1), which will be also solutions of the problem (1.1), can be expressed in the following form in each of the intervals ξ_k (3.3)

$$w(x,t) = |X(x)|\, e^{pt}\cos[\psi(x) + qt], \qquad w(x,t) = |X(x)|\, e^{pt}\sin[\psi(x) + qt]$$

Using the relations (1.2) and (2.2) we can prove the following assertion when the condition $0 < v^2 < 1$ is satisfied, all the eigenvalues ω of the problem (3.2) are purely imaginary numbers ($\omega = iq,\ q \neq 0,\ p = 0$), upon which the condition $\psi(x) \neq \text{const}$ is satisfied for the corresponding solutions (3.3) in each of the intervals ξ_k, i.e. the vibrations (3.3) have the form of waves, running along the pipe.

Let us note, that in the case when the condition $0 < v^2 < 1$ is violated, the assertion stated above may not be satisfied. For example, when any of the conditions (2.3) is fulfilled, the generalized boundary-value problem (3.2) has a bouble eigenvalue $\omega = 0$, to which corresponds the eigenfunction $X(x) = \sin m\pi x$ and the adjoint [5] function $\varphi(x)$, which is contained in the solution (2.4).

The condition $\psi = \text{const}$ is satisfied ($\psi = 0$ or $\psi = \pi$) for the eigenfunction $X(x) = \sin m\pi x$ on each of the intervals ξ_k where $X(x)$ does not vanish.

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Translated by E.Z.